

Section 11.4. The Comparison Tests.

The idea here is to compare a given series with a series that is known to converge or diverge (usually geometric series $\sum a_n n^n$ or p-series $\sum \frac{1}{n^p}$, since they are easy to judge). For example, the series $\sum_{n=1}^{\infty} \frac{1}{3^{n+2}}$ reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$, a geometric series with ratio $\frac{1}{3} < 1$. We know $\sum \left(\frac{1}{3}\right)^n$ converges, and has the sum $\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$. Moreover, $\frac{1}{3^{n+2}} < \frac{1}{3^n}$ for every $n \geq 1$. This means that the n^{th} term of $\sum \frac{1}{3^{n+2}}$ is less than the n^{th} term of $\sum \frac{1}{3^n}$. Thus, $\sum \frac{1}{3^{n+2}}$ should have a smaller sum than $\sum \frac{1}{3^n}$. Since $\sum \frac{1}{3^n}$ has a finite sum, then so does $\sum \frac{1}{3^{n+2}}$, and the latter series is convergent as well! We generalize this concept in the "Comparison (direct) Test": Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. Then

- (i) if $\sum b_n$ converges, and $a_n \leq b_n$ for all n , then $\sum a_n$ converges;
- (ii) if $\sum b_n$ diverges, and $a_n \geq b_n$ for all n , then $\sum a_n$ diverges.

Example ① Test $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{7}{2}}}$ for convergence or divergence. Let $a_n = \frac{1}{n^{\frac{7}{2}}}$, $b_n = \frac{1}{n^7}$. Then $a_n > 0$ and $b_n > 0$ for all $n \geq 1$. Moreover $\sum b_n = \sum \frac{1}{n^7}$ converges by p-series, $p = 7 > 1$. Since $a_n = \frac{1}{n^{\frac{7}{2}}} < b_n = \frac{1}{n^7}$, we conclude that $\sum a_n = \sum \frac{1}{n^{\frac{7}{2}}}$ converges by direct comparison to $\sum \frac{1}{n^7}$.

② Is $\sum_{n=1}^{\infty} \frac{4 + \cos(n)}{n}$ convergent or divergent? Let $a_n = \frac{4 + \cos(n)}{n}$, $b_n = \frac{1}{n}$. Then $a_n > 0$ and $b_n > 0$ for all $n \geq 1$ (since $\cos(n) > -1$), and $\frac{4 + \cos(n)}{n} > \frac{1}{n}$ for $n \geq 1$. Since $\sum \frac{1}{n}$ diverges by p-series, $p = 1$, then $\sum a_n$ diverges as well by direct comparison to $\sum \frac{1}{n}$.

③ is $\sum_{n=3}^{\infty} \frac{\ln(n)}{n^2}$ convergent or divergent? (Hint: $\ln(n) < \sqrt{n}$ for all $n > 0$)

Let $a_n = \frac{\ln(n)}{n^2}$, and $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$; then $a_n > 0$ and $b_n > 0 \forall n \geq 3$; also $a_n < b_n$ for all $n \geq 3$ since $\ln(n) < \sqrt{n}$ for all $n > 0$. Since $\sum b_n$ converges by p-series, $p = \frac{3}{2} > 1$, then $\sum a_n$ converges by direct comparison to $\sum \frac{1}{n^{3/2}}$.

④ We cannot use the direct comparison test (DCT) on the series $\sum_{n=1}^{\infty} \frac{n}{5-n^3}$ since it has negative terms!

The limit comparison test (L.C.T): suppose $\sum a_n, \sum b_n$ are series with positive terms. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$, then either both series converge, or both series diverge.

Example ⑤ is $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ convergent or divergent? Let $a_n = \frac{1}{2^{n-1}}, b_n = \frac{1}{2^n}$. Then $a_n > 0, b_n > 0$ for all $n \geq 1$, and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2^{n-1}}}{\frac{1}{2^n}} = 1$. Since $0 < 1 < \infty$, we may use the L.C.T. Since $\sum b_n$ converges (geometric series, $r = \frac{1}{2} < 1$), then so does $\sum a_n$, by the L.C.T.

⑥ Test $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ for convergence or divergence. For large n , most contribution in the numerator comes from $2n^2$, and in the denominator from $\sqrt{n^5} = n^{5/2}$. So we compare to $\frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$. Let $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}, b_n = \frac{2}{n^{1/2}}$; Then $a_n, b_n > 0$ for all $n \geq 1$, and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ (show work) and $0 < 1 < \infty$. So we may use the L.C.T. Since $\sum b_n = \sum \frac{2}{n^{1/2}}$ diverges by p-series ($p = \frac{1}{2} < 1$), then so does $\sum a_n$ by the L.C.T.

⑦ What do you the following series to, if you were using the LCT?

a) $\sum_{n=1}^{\infty} a_n$, where $a_n = (5+5n)^{-2} = \frac{1}{(5+5n)^2}$. At ∞ , a_n behaves like $\frac{1}{25n^2}$.

So we compare $\sum_{n=1}^{\infty} a_n$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

b) $\sum_{n=1}^{\infty} a_n$, where $a_n = \left(\frac{3n^2 + 2n + 5}{7n^4 + 7n + 5\sqrt{n}} \right)^4$. The dominant terms in the numerator and denominator are, respectively, $3n^2$ and $7n^4$. So, at ∞ , a_n acts like $\left(\frac{3n^2}{7n^4} \right)^4 = \left(\frac{3}{7} \right)^4 \cdot \frac{1}{n^8}$. Thus, we may compare to $\sum_{n=1}^{\infty} \frac{1}{n^8}$.

c) $\sum_{n=1}^{\infty} a_n$, where $a_n = (7+6^n)^{-3}$. At ∞ , a_n acts like $(6^n)^{-3} = \left(\frac{1}{6^3} \right)^n$.

So compare to $\sum_{n=1}^{\infty} b_n$, where $b_n = \left(\frac{1}{6^3} \right)^n$; here $a_n > 0$, $b_n > 0$ for $n \geq 1$,

and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{6^n}{7+6^n} \right)^3 = 1$. Since $0 < 1 < \infty$, $\sum a_n$ converges

by the L.C.T applied to $\sum \frac{1}{(6^3)^n}$, a geometric series with ratio $\frac{1}{6^3} < 1$.

d) $\sum_{n=1}^{\infty} a_n$, where $a_n = \left(\frac{3n^2 + 3n + 7^{\frac{9}{10}n}}{18^{n+2} + 6n + 6\sqrt{n}} \right)^8$; at ∞ , a_n acts like $\left(\frac{7^{\frac{9}{10}n}}{18^2} \right)^8 = \left(\frac{1}{18^2} \right)^8 \cdot \left[\left(\frac{7^{\frac{9}{10}}}{18} \right)^8 \right]^n$. So we compare to $\sum_{n=1}^{\infty} \left[\left(\frac{7^{\frac{9}{10}}}{18} \right)^8 \right]^n$ (geometric).